Conditional limits of W_p scale mixture distributions

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Abstract: In this paper we introduce the class of W_p scale mixture random vectors with a particular radial decomposition and a independent splitting property specified by some random variable $W_p, p \in (0, \infty)$. We derive several conditional limit results assuming that the distribution of the random radius is in the max-domain of attraction of a univariate extreme value distribution and W_p has a certain tail asymptotic behaviour. As an application we obtain the joint asymptotic distribution of concomitants of order statics considering certain bivariate W_p scale mixture random samples.

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1 Introduction

Let (X, Y) be a bivariate spherical random vector with stochastic representation

$$(X, Y) \stackrel{d}{=} (RU_1, RU_2), \tag{1.1}$$

where R is a positive random variable with distribution function F being independent of the bivariate random vector (U_1, U_2) which uniformly distributed on the unit circle of $I\!\!R^2$ ($\stackrel{d}{=}$ stands for equality of distribution functions). The radial decomposition (1.1) makes the spherical vectors a very tractable class with well-known properties (see e.g. Kotz (2000)). An interesting distributional result is that if we condition on one component and consider, say $Z_u^* := (Y|X = u)$, then the distribution function of Z_u^* can be easily specified in terms of F. A canonical example is when $R^2 \stackrel{d}{=} X^2$, with X and Y two independent standard Gaussian random variables (with mean 0 and variance 1). In this model $Z_u^*, u \in I\!\!R$ is a standard Gaussian random variable. Remarkably, even though the random variable $Z_u := (Y|X > u)$ is not a Gaussian one, when u tends to ∞ it can be approximated (in distribution) by a standard Gaussian random variable. In fact, interesting enough, both Z_u and Z_u^* can be approximated in distribution by a Gaussian random variable, provided that the distribution function F is in the Gumbel max-domain of attraction (see Berman (1983), Hashorva (2006)). Two other important instances are when F is in the Fréchet and Weibull max-domain of attraction (see Berman (1992)), since again Z_u, and Z_u^* can be approximated by some random variables with known distribution (not dependent on F).

The radial decomposition (1.1) suggests that conditional limit results could be derived without imposing specific distributional assumptions on the random vector (U_1, U_2). The first attempt in this direction is made in Hashorva (2009) where the approximating distribution is in general no longer Gaussian but a polar Kotz distribution.

The main interest related to conditional limit results is due to their important role in the modeling of rare events, see e.g., Berman (1992), Joshi and Nagaraja (1995), Ledford and Twan (1998), Abdous et al. (2005), Heffernan and Resnick (2007), Balkema and Embrechts (2007), or Resnick (2007).

Recent theoretical developments with interesting statistical applications concerning conditional extreme value models are presented (bivariate setup) in the recent deep contributions Heffernan and Resnick (2007) and Das and Resnick (2008a,b).

In this paper we are interested in multivariate distributions with radial decomposition for which conditional limit results such as the Kotz approximation hold under general asymptotic settings. Borrowing the idea of Hashorva (2008a) where beta-independent random vectors are discussed, we introduce in this paper a new class of random vectors with prominent members the Dirichlet and the elliptical random vectors.

Without going to mathematical details we briefly state the main contributions of this paper:

a) First the class of W_p scale mixture random vectors is introduced; for this model we show a tractable stochastic representation of conditional random vectors (see Theorem 2.2 below). This representation, which is of some interest on its own, is the key to our asymptotic results.

b) Under certain asymptotic restrictions on the class of W_p scale mixture random vectors we obtain several conditional limit results extending those presented in Hashorva (2008a) for the class of beta-independent random vectors.

c) Applying the Kotz approximation of W_p scale mixture random vectors we derive the joint asymptotic distribution of the concomitants of order statistics (bivariate setup). It is interesting that for our model, which includes as the special case the bivariate elliptical one, the concomitants of order statistics are asymptotically independent.

Three other applications of our results (not developed here) concern i) joint tail asymptotics of bivariate W_p scale mixture random vectors (see Hashorva (2008a)), ii) the asymptotic independence and asymptotic behaviour of extremes of W_p scale mixture random sequences, and iii) the estimation of conditional distributions and conditional quantile functions in multivariate W_p scale mixture models (see Abdous et al. (2008), Hashorva (2008a,2009)).

Organisation of the paper: The main results are given in Section 3. In Section 4 we present the application to concomitants of order statistics. Proofs of the stated results are relegated to Section 5.

2   Preliminaries

We introduce first some standard notation. Throughout this paper I is a non-empty index subset of $\{1, \ldots, d\}, d \geq 2$, with $m := |I| < d$ elements. For given $\boldsymbol{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ we define the subvectors $\boldsymbol{x}_I := (x_i, i \in I)^\top, \boldsymbol{x}_J := (x_i, i \in J)^\top$, with $J := \{1, \ldots, d\} \setminus I$ (here $^\top$ denotes the transpose sign). Similarly, given a matrix $A \in \mathbb{R}^{d \times d}$ we define $A_{II}, A_{IJ}, A_{JI}, A_{JJ}$ for its submatrices obtained by keeping the rows and columns with the indices in the corresponding index sets. If $(A_{JJ})^{-1}$ exists we write instead A_{JJ}^{-1}. Next, let $\boldsymbol{y} = (y_1, \ldots, y_d)^\top$ be another vector in $\mathbb{R}^d$ and let $\|\cdot\|_J$ be a norm in $\mathbb{R}^{d-m}$. We define

$$\boldsymbol{x} + \boldsymbol{y} := (x_1 + y_1, \ldots, x_d + y_d)^\top,$$

$$\boldsymbol{x} > \boldsymbol{y}, \text{ if } x_i > y_i, \quad \forall i = 1, \ldots, d,$$

$$\boldsymbol{x} \geq \boldsymbol{y}, \text{ if } x_i \geq y_i, \quad \forall i = 1, \ldots, d,$$

$$\boldsymbol{x} \neq \boldsymbol{y}, \text{ if for some } i \leq d \ x_i \neq y_i,$$

$$\boldsymbol{ax} := (a_1 x_1, \ldots, a_d x_d)^\top, \quad c\boldsymbol{x} := (cx_1, \ldots, cx_d)^\top, \quad \boldsymbol{a} \in \mathbb{R}^d, c \in \mathbb{R},$$

$$\|\boldsymbol{x}_J\|_A := \|A_{JJ}^{-1} \boldsymbol{x}_J\|_J, \quad A \in \mathbb{R}^{d \times d}.$$

In order to simplify the notation, we write $\boldsymbol{x}_I^\top$ instead of $(\boldsymbol{x}_I)^\top$, respectively.

Given $\boldsymbol{Z}$ a random vector with distribution function G and density function g we write alternatively $\boldsymbol{Z} \sim G$, and $\boldsymbol{Z} \simeq g$, respectively. When G is a univariate distribution we denote by $\overline{G}$ its survivor function and by x_G its upper endpoint. In the following $Z \sim Beta(a, b)$ or $Z \sim Gamma(a, b)$ mean that Z is Beta or Gamma distributed with positive parameters a, b, respectively. The corresponding density functions are

$$x^{a-1}(1-x)^{b-1}\Gamma(a+b)/(\Gamma(a)\Gamma(b)), x \in (0, 1), \quad \text{and } x^{a-1} \exp(-bx)b^a/\Gamma(a), x \in (0, \infty),$$

with $\Gamma(\cdot)$ the Gamma function.

Let $I, p \in (0, \infty)$ be given and throughout this paper $\boldsymbol{U}$ is a random vector in $\mathbb{R}^d, d \geq 2$ with specific properties defined in terms $\|\cdot\|_I, \|\cdot\|_J$ two given norms in $\mathbb{R}^m$ and $\mathbb{R}^{d-m}$, respectively. Explicitly, we suppose that

$$\|\boldsymbol{U}_I\|_I = \|\boldsymbol{U}_J\|_J = 1$$

almost surely and $\boldsymbol{U}_I$ is independent of $\boldsymbol{U}_J$. In our definition below, the properties of $\boldsymbol{U}$ are related to p and a random variable $W_p \in [0, 1]$.

We arrive at the following definition:

Definition 2.1. *Let $R \sim F, W_p \sim G$ be two random variables such that $R > 0, W_p \in [0, 1]$ almost surely. We define a W_p scale mixture random vector $\boldsymbol{X}$ in $\mathbb{R}^d, d \geq 2$ via the stochastic representation*

$$\boldsymbol{X} \stackrel{d}{=} A\boldsymbol{S}, \quad \boldsymbol{S}_I \stackrel{d}{=} RW\boldsymbol{U}_I, \quad \boldsymbol{S}_J \stackrel{d}{=} RW_p\boldsymbol{U}_J, \quad W := (1 - W_p^p)^{1/p}, \tag{2.1}$$

where $A \in \mathbb{R}^{d \times d}$, and $R, W, \boldsymbol{U}_I, \boldsymbol{U}_J$ are mutually independent.

Clearly, the distribution function of $\boldsymbol{X}$ with stochastic representation (2.1) is determined by A, F, G, I, p and $\boldsymbol{U}_I, \boldsymbol{U}_J$. In the following we refer to $\boldsymbol{X}$ as a W_p scale mixture random vector with parameters A, F, G, I, p, or simply as a W_p scale mixture random vector. If G possesses a density function g (which we assume in the following) we mention g instead of G. Both $\boldsymbol{U}_I$ and $\boldsymbol{U}_J$ are also important, however since we deal with the same $\boldsymbol{U}$ we do not consider these random vectors as further parameters in our definition.

In the special case that $W_p^\delta, \delta \in (0, \infty)$ is a Beta distributed random variable and

$$\|\boldsymbol{x}_I\|_I := \left(\sum_{i \in I} |x_i|^p\right)^{1/p}, \quad \|\boldsymbol{x}_J\|_J := \left(\sum_{i \in J} |x_i|^p\right)^{1/p}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

$\boldsymbol{X}$ is referred to as a beta-independent random vector (see Hashorva (2008a)).

Any spherical random vector $\boldsymbol{X}$ in $\mathbb{R}^d$ with positive associated random radius satisfies (2.1) with $W^2 \sim Beta(m/2, (d-m)/2)$ and $\boldsymbol{U}_I, \boldsymbol{U}_J$ two independent random vectors being uniformly distributed on the unit spheres of $\mathbb{R}^m$ and $\mathbb{R}^{d-m}$, respectively ($\|\cdot\|_I$ and $\|\cdot\|_J$ are the L_2-norms in the corresponding spaces). Also Dirichlet and beta-independent random vectors belong to the class of W_p scale mixture random vectors.

We impose further the following assumption on $\boldsymbol{X}$ needed in the definition of the conditional random vector $\boldsymbol{X}_I | \boldsymbol{X}_J$.

A1. If $\boldsymbol{X}$ is a random vector with stochastic representation (2.1) and $K \subset J$ where $J \setminus K$ has only one element, then $\boldsymbol{U}_K$, possesses a positive density function defined for all $\boldsymbol{u}_K \in \mathbb{R}^{|K|}$ with $\boldsymbol{u} \in \mathbb{R}^d$ such that $\|\boldsymbol{u}_J\| = 1$. When J has

only one element, then we suppose that $P\{U_J = 1\} > 0$. Further, we suppose that A_{JJ}^{-1} exits.

In the bivariate setup $d = 2$ the random vector (X, Y) defined by

$$X = RI_1 W, \quad Y = \rho RI_1 W + (1 - \rho^p)^{1/p} RI_2 W_p, \quad p \in (0, \infty), \quad \rho \in (-1, 1), \tag{2.2}$$

where $R > 0, W := (1 - W_p^p)^{1/p} \in [0, 1]$ almost surely and I_1, I_2 assume values in $\{-1, 1\}$ with $P\{I_1 = 1\}P\{I_2 = 1\} \in (0, 1]$ is a W_p scale mixture random vector, provided that I_1, I_2, R, W are mutually independent.

Let $g : (0, 1) \to (0, \infty)$ be the density function of W_p, and define the distribution function $Q_{F,g,\tau}$ by

$$Q_{F,g,\tau}(z) := 1 - \frac{\int_{(\tau^p + z^p)^{1/p}}^{x_F} g(\tau/r)\frac{1}{r}\, dF(r)}{\int_\tau^{x_F} g(\tau/r)\frac{1}{r}\, dF(r)}, \quad \forall z \in (0, (x_F{}^p - \tau^p)^{1/p}), \quad \tau \in (0, x_F). \tag{2.3}$$

We have the following result:

Theorem 2.2. *Let X be a W_p scale mixture random vector in $\mathbb{R}^d, d \geq 2$ with representation (2.1) and parameters A, F, g, I, p. Assume that A_{JI} has all entries equal 0, and Assumption A1 is satisfied. If $a \in \mathbb{R}^d$ is such that $F(\|a_J\|_A) \in (0, 1)$, then we have the stochastic representation*

$$\left(X_I \Big| X_J = a_J\right) \quad \overset{d}{=} \quad A_{II} R_{F,g,\|a_J\|_A} U_I + A_{IJ} A_{JJ}^{-1} a_J, \tag{2.4}$$

with U_I independent of the positive random variable $R_{F,g,\|a_J\|_A}$ which has distribution function $Q_{F,g,\|a_J\|_A}$.

Remarks: *(a) If F in Theorem 2.2 possesses a distribution function f, then also $Q_{F,g,\|a_J\|_A}$ possesses a density function given by (set $c := \|a_J\|_A \in (0, x_F)$)*

$$q_{f,g,c}(z) := \frac{z^{p-1}}{c^p + z^p} \frac{g(c/(c^p + z^p)^{1/p})f((c^p + z^p)^{1/p})}{\int_c^{x_F} g(c/s)\frac{1}{s}f(s)\, ds}, \quad \forall z \in (0, (x_F{}^p - c^p)^{1/p}). \tag{2.5}$$

(b) When X is an elliptical random vector, then (2.4) holds for any $I \subset \{1, \ldots, d\}$ without the assumption that A_{JI} has all entries equal 0.

(c) Examples of norms appearing in the definition of the random vector U are

$$\|x_I\|_I := \left(\sum_{i \in I} |x_i|^{q_1}\right)^{1/q_1}, \quad \|x_J\|_J := \left(\sum_{i \in J} |x_i|^{q_2}\right)^{1/q_2}, \quad x \in \mathbb{R}^d, \quad q_1, q_2 \in [1, \infty). \tag{2.6}$$

We note in passing that our results can be stated also when $q_1, q_2 \in (0, 1)$.

The main asymptotic condition imposed on the distribution function F is that it belongs to the max-domain of attraction of a univariate extreme value distribution function H. Explicitly, we suppose that

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| F^n(a_n x + b_n) - H(x) \right| = 0 \tag{2.7}$$

holds for some $a_n > 0, b_n \in \mathbb{R}, n \geq 1$. The distribution function H is either the unit Gumbel distribution $\Lambda(x) := \exp(-\exp(-x)), x \in \mathbb{R}$, the unit Weibull distribution $\Psi_\gamma(x) := \exp(-|x|^\gamma), \gamma \in (0, \infty), x \in (-\infty, 0)$, or the unit Fréchet distribution $\Phi_\gamma(x) := \exp(-x^{-\gamma}), \gamma \in (0, \infty), x \in (0, \infty)$. See Reiss (1989), Embrechts et al. (1997), Falk et al. (2004), De Haan and Ferreira (2006), or Resnick (2008) for details on univariate extreme value distributions and max-domains of attraction.

3 Kotz Approximation

Consider X a W_p scale mixture random vector in $\mathbb{R}^d, d \geq 2$ with representation (2.1) and parameters A, F, g, I, p, and let $u_n, n \geq 1$ be a sequence of constants in $\mathbb{R}^d$ satisfying $\|u_{n,J}\|_A \in (0, x_F), n \geq 1$. Next, we introduce two sequences

of random vectors $\boldsymbol{Z}_n, n \geq 1$, and $\boldsymbol{Z}_n^*, n \geq 1$ defined in the same probability space satisfying

$$\boldsymbol{Z}_{n,I} \overset{d}{=} (\boldsymbol{X}_I - A_{IJ}A_{JJ}^{-1}\boldsymbol{u}_{n,J})\Big|\boldsymbol{X}_J = \boldsymbol{u}_{n,J}, \quad \boldsymbol{Z}_n^* \overset{d}{=} \left((\boldsymbol{X}_I - \boldsymbol{u}_{n,J}A_{IJ}), (\boldsymbol{X}_J - \boldsymbol{u}_{n,J})\right)\Big|\boldsymbol{X}_J > \boldsymbol{u}_{n,J}, \quad n \geq 1. \tag{3.1}$$

For notational simplicity we write $\boldsymbol{x}_{n,K}$ instead of $(\boldsymbol{x}_n)_K$ for some $\boldsymbol{x}_n \in \mathbb{R}^d$ and $K \subset \{1, \ldots, d\}$.

Our main concern in this section is the asymptotic approximation of these sequences when $\|\boldsymbol{u}_{n,J}\|_A \to x_F$ assuming further that F is in the Gumbel max-domain of attraction satisfying (2.7) with $H = \Lambda$. The latter assumption (henceforth abbreviated $F \in MDA(\Lambda, w)$ is equivalent with (see e.g., Embrechts et al. (1997))

$$\lim_{t \uparrow x_F} \frac{\overline{F}(t + x/w(t))}{\overline{F}(t)} = \exp(-x), \quad \forall x \in \mathbb{R}, \tag{3.2}$$

where w is some positive scaling function. If $\boldsymbol{X}$ is an elliptical random vector i.e., $\boldsymbol{U}$ is uniformly distributed on the unit sphere of $\mathbb{R}^d$ (with respect to L_2-norm), then in view of Hashorva (2006) both conditional random vectors $\boldsymbol{Z}_{n,I}$ and $\boldsymbol{Z}_{n,I}^*$ can be approximated in distribution by a Gaussian random vector, provided that $\|\boldsymbol{u}_{n,J}\|_A \to x_F$.

We note in passing that the Gaussian approximation for bivariate elliptical random vectors is first obtained in full generality in Berman (1983). In Hashorva (2008a) it is shown that the limiting random vector is a Kotz Type I polar random vector if $\boldsymbol{X}$ is a beta-independent random vector. In our definition $\boldsymbol{Y} \in \mathbb{R}^k, k \geq 2$ is referred to as a Kotz Type I scale mixture random vector if $\boldsymbol{Y} \overset{d}{=} BR\boldsymbol{V}$, where $B \in \mathbb{R}^{k \times k}$ and $R^q \sim Gamma(\alpha, \beta), \alpha, \beta, q \in (0, \infty)$. Further R is positive and independent of $\boldsymbol{V} \in \mathbb{R}^k$ and for some norm $\|\cdot\|_k$ in $\mathbb{R}^k$ we have $\|\boldsymbol{V}\|_k = 1$ almost surely. When $\|\cdot\|_k$ is the Euclidian norm, $\boldsymbol{X}$ is referred to as a Kotz Type I polar random vector.

In this section we show that Kotz approximation of conditional random vectors $\boldsymbol{Z}_{n,I}$ and $\boldsymbol{Z}_{n,I}^*$ holds also for the general settings of W_p scale mixture random vectors. Instead of some distributional assumptions on $\boldsymbol{U}$, we suppose only that the density function g of W_p satisfies

$$\lim_{u \to \infty} \frac{g(1 - x/u)}{g(1 - 1/u)} = x^{\alpha - 1}, \quad \forall x > 0 \tag{3.3}$$

for some $\alpha \in (0, \infty)$, i.e., g is regularly varying at 1 with index $\alpha - 1$. As will be shown below the parameter α together with p determines the conditional limit distribution.

We state next the main result of this section.

Theorem 3.1. *Let I be an index set of $\{1, \ldots, d\}, d \geq 2$, and let $\boldsymbol{X}$ be a W_p scale mixture random vector in $\mathbb{R}^d$ with representation (2.1) and parameters A, F, g, I, p satisfying Assumption A1. Suppose that (3.3) holds with $\alpha \in (0, \infty)$ and for any $\varepsilon \in (0, 1)$*

$$g(x) \leq c_\varepsilon x^{\gamma_\varepsilon}, \quad \forall x \in (0, \varepsilon) \tag{3.4}$$

is valid with $c_\varepsilon \in (0, \infty), \gamma_\varepsilon \in \mathbb{R}$. Assume further that $F \in MDA(\Lambda, w)$ and A_{JI} has all entries equal 0.

a) Let $\boldsymbol{u}_n, n \geq 1$ be constants in $\mathbb{R}^d$ such that $F(\|\boldsymbol{u}_{n,J}\|_A) \in (0, 1), n \geq 1$ and $\lim_{n \to \infty} \|\boldsymbol{u}_{n,J}\|_A = x_F$. Then we have the convergence in distribution

$$h_n \boldsymbol{Z}_{n,I} \overset{d}{\to} A_{II} \mathcal{R}_\alpha \boldsymbol{U}_I, \quad n \to \infty, \tag{3.5}$$

where $h_n := (\|\boldsymbol{u}_{n,J}\|_A^{1-p} w(\|\boldsymbol{u}_{n,J}\|_A))^{1/p}, n > 1$, and $\mathcal{R}_\alpha$ is a positive random variable independent of $\boldsymbol{U}_I$ such that $\mathcal{R}_\alpha^p \sim Gamma(\alpha, 1/p)$.

b) Let $\boldsymbol{u}_n := u_n(1, \ldots, 1)^\top \in \mathbb{R}^d, u_n \in (0, x_F), n \geq 1$ be such that $\lim_{n \to \infty} u_n = x_F$. If $I = \{1, \ldots, d-1\}, J = \{d\}, A_{JJ} = 1$, and A_{IJ} has all entries equal 0 if $p \in (0, 1)$, then we have (set $1_p := 1$ if $p = 1$, and $1_p := 0$ otherwise)

$$\left(h_n \boldsymbol{Z}_{n,I}^*, w(u_n) \boldsymbol{Z}_{n,J}^*\right) \overset{d}{\to} \left(A_{II} \mathcal{R}_\alpha \boldsymbol{U}_I + \mathcal{E} 1_p A_{IJ}, \mathcal{E}\right), \quad n \to \infty, \tag{3.6}$$

with $\mathcal{E} \sim Gamma(1, 1)$ independent of $(\mathcal{R}_\alpha, \boldsymbol{U}_I)$.

Remark 3.2. *(a) The random vector $\boldsymbol{Y}_I := A_{II}\mathcal{R}_\alpha\boldsymbol{U}_I$ appearing in (3.5) is a Kotz Type I scale mixture random vector. Therefore we refer to the distribution approximation in (3.5) as the Kotz approximation. It is well-known (see e.g., Kotz et al. (2000)) that $\boldsymbol{Y}_I$ is a Gaussian random vector in $\mathbb{R}^m, m := |I|$ with covariance matrix $A_{II}A_{II}^\top$, provided that $p = 2, \alpha = m/2$, and $\boldsymbol{U}_I$ is uniformly distributed on the unit sphere of $\mathbb{R}^m$ (with respect to L_2-norm). (b) The Kotz approximation above is stated in terms of convergence of distribution functions. It is of some interest to strengthen this to convergence of the corresponding density functions, which we refer to as the strong Kotz approximation. The random vector $\boldsymbol{Z}_{n,I}, \boldsymbol{Z}_{n,I}^*, n \geq 1$ possess a density function (recall (2.5)) if both R and $\boldsymbol{U}_I$ possess a positive density function. If $R \backsimeq f$ such that*

$$\lim_{u\uparrow x_F} \frac{f(u + x/w(u))}{f(u)} = \exp(-x), \quad \forall x \in \mathbb{R} \tag{3.7}$$

holds with some positive scaling function w, then for $\boldsymbol{X}$ a L_p Dirichlet random vector (see Hashorva and Kotz (2009)) the convergence in (3.6) can be strengthened to the strong Kotz approximation. With similar arguments as in the aforementioned paper utilising further (2.5), it follows that both (3.5) and (3.6) can be strengthened to the local uniform convergence of the corresponding density functions, provided that (3.7) is satisfied and $\boldsymbol{U}_I$ possesses a positive density function.

We present next two illustrating examples.

Example 1. [Kotz Type III W_p scale mixture] We refer to $\boldsymbol{X}$ in $\mathbb{R}^d, d \geq 2$ as a Kotz Type III W_p scale mixture random vector if it has stochastic representation (2.1) (for some given index set I) with $A \in \mathbb{R}^{d\times d}$ such that A_{JJ} is non-singular, and

$$\overline{F}(u) = (1 + o(1))Ku^N \exp(-ru^\delta), \quad K > 0, \delta > 0, N \in \mathbb{R}, \quad u \to \infty. \tag{3.8}$$

Assume further that W_p possesses the positive density function g which is bounded in $[0, 1]$ and for any $\varepsilon \in (0, 1)$

$$g(u) = c(u)(1 - u)^{\alpha-1}, \quad \forall u \in (\varepsilon, 1), \quad \alpha \in (0, \infty),$$

with $c(u)$ some positive measurable function such that $\lim_{u\uparrow1} c(u) = c \in (0, \infty)$. Clearly g satisfies the assumptions of Theorem 3.1. Since F is in the Gumbel max-domain of attraction with scaling function $w(s) = r\delta s^{\delta-1}, s > 0$ the aforementioned theorem implies for any sequence $\boldsymbol{u}_n, n \geq 1$ satisfying $\lim_{n\to\infty}\|\boldsymbol{u}_{n,J}\|_A = \infty$

$$(r\delta)^{1/p}\|\boldsymbol{u}_{n,J}\|_A^{\delta/p-1}\boldsymbol{Z}_{n,I} \xrightarrow{d} A_{II}\mathcal{R}_\alpha\boldsymbol{U}_I, \quad n \to \infty,$$

where $\mathcal{R}_\alpha$ is a positive random variable independent of $\boldsymbol{U}_I$ such that $\mathcal{R}_\alpha^p \sim Gamma(\alpha, 1/p), \mathcal{R}_\alpha \in (0, \infty)$.

Example 2. [F with finite upper endpoint] Let $\boldsymbol{X} = AR\boldsymbol{U}$ be a W_p scale mixture random vector in $\mathbb{R}^d, d \geq 2$. Suppose that $R \sim F$ with

$$\overline{F}(u) = (1 + o(1))c_1 \exp(-c_2(x_F - u)^{-\lambda}), \quad c_1, c_2, \lambda \in (0, \infty) \quad u \uparrow x_F \in (0, \infty).$$

It follows easily that $F \in MDA(\Lambda, w)$ with $w(s) = c_2\lambda(x_F - s)^{-\lambda-1}, s \in (0, 1)$. Consequently, if $A, I, \boldsymbol{U}, \boldsymbol{u}_n, n \geq 1$ are such that the assumptions of Theorem 3.1 are satisfied, then we have the approximation

$$\frac{(c_2\lambda)^{1/p}x_F^{1/p-1}}{(x_F - \|\boldsymbol{u}_{n,J}\|)^{(\lambda+1)/p}}\boldsymbol{Z}_{n,I} \xrightarrow{d} A_{II}\mathcal{R}_\alpha\boldsymbol{U}_I, \quad n \to \infty,$$

where $\lim_{n\to\infty}\|\boldsymbol{u}_{n,J}\| = x_F$ and $\mathcal{R}_\alpha, \boldsymbol{U}_I$ are as in Example 1.

4 Regularly Varying $\overline{F}$

In this section we deal with distribution functions F in the Weibull or Fréchet max-domain of attraction. Specifically, in the former case

$$\lim_{u \to \infty} \frac{\overline{F}(1 - x/u)}{\overline{F}1 - 1/u)} = x^\gamma, \quad \forall x > 0, \quad \gamma \in (0, \infty) \tag{4.1}$$

is valid for some $\gamma \in (0, \infty)$. Condition (4.1) is equivalent with (2.7) where $H = \Psi_\gamma$ is the unit Weibull distribution. When F is in the max-domain of attraction of the Fréchet distribution Φ_γ, then we have

$$\lim_{u \to \infty} \frac{\overline{F}(xu)}{\overline{F}(u)} = x^{-\gamma}, \quad \forall x > 0. \tag{4.2}$$

Under these assumptions, where the survivor function $\overline{F}$ is regularly varying at the upper endpoint, the approximation of the conditional distribution of W_p scale mixture random vectors can be carried out as in the case of beta-independent random vectors.

It is interesting that when (3.3) and (4.1) hold, then the conditional limit distributions are completely specified by α, p and γ. For the Fréchet case we do not to impose any asymptotic assumptions on g, therefore the conditional limit distribution depends only on the regularly varying index γ. Both α and p do not appear in the asymptotics.

We state next the main result of this section.

Theorem 4.1. *Let $\mathbf{X}, \mathbf{u}_n, \mathbf{Z}_{n,I}, n \geq 1$, be as in Theorem 3.1, and let $\mathcal{R}_{\alpha,\gamma}, -\mathcal{E}$ be two independent positive random variables such that $\mathcal{R}_{\alpha,\gamma}^p \sim Beta(\alpha, \gamma)$, and $\mathcal{E} \sim 1 - |x|^{\alpha+\gamma}, x \in (-1, 0)$ being both independent of $\mathbf{U}_I$. Assume that F satisfies (4.1), and let $a_n, n \geq 1$ be positive constants such that $\lim_{n \to \infty} a_n = 0$.*
a) If for all large n we have $\|\mathbf{u}_{n,J}\|_A = 1 - a_n$, then (set $h_n := (pa_n)^{-1/p}, n \geq 1$)

$$h_n \mathbf{Z}_{n,I} \xrightarrow{d} A_{II} \mathcal{R}_{\alpha,\gamma} \mathbf{U}_I, \quad n \to \infty. \tag{4.3}$$

b) If $I = \{1, \ldots, d-1\}, A_{JJ} = 1$, and A_{IJ} has all entries equal 0 if $p \in (0, 1)$, then for any $\mathbf{x} \in \mathbb{R}^d$ with $x_d \in (-\infty, 0)$

$$\lim_{n \to \infty} \mathbf{P}\Big\{h_n(\mathbf{X}_I - A_{IJ}) \leq \mathbf{x}_I, (X_d - 1)/a_n \leq x_d \big| X_d > 1 - a_n\Big\}$$
$$= \mathbf{P}\{|\mathcal{E}|^{1/p}(A_{II}\mathcal{R}_{\alpha,\gamma}\mathbf{U}_I - 1_p A_{IJ}) \leq \mathbf{x}_I, \mathcal{E} \leq x_d\} \tag{4.4}$$

is valid.
c) If (4.2) holds, then for any $\mathbf{a} \in \mathbb{R}^d$ such that $\|\mathbf{a}_J\|_A > 0$ we have (set $\mathbf{u}_n := \mathbf{a}/a_n, n \geq 1$)

$$a_n \mathbf{Z}_{n,I} \xrightarrow{d} A_{II} \mathcal{R} \mathbf{U}_I, \quad n \to \infty, \tag{4.5}$$

with $\mathcal{R} > 0$ independent of $\mathbf{U}_I$ satisfying $\mathcal{R} \sim Q_{M,g,\|\mathbf{a}_J\|_A}$, where $M(s) := 1 - \|\mathbf{a}_J\|_A^\gamma s^{-\gamma}, \forall s \geq \|\mathbf{a}_J\|_A$.

Remark 4.2. *(a) The convergence in (4.3) and (4.5) is stated in Hashorva et al. (2007) for L_p Dirichlet random vectors, and Hashorva (2008a) for beta-independent random vectors.*
(b) Under von Misses conditions on the density function f of F the above asymptotics can be strengthened to local uniform convergence of the corresponding density functions.
(c) Condition (4.1) is satisfied for instance for the Beta distribution, whereas condition (4.2) is satisfied by distributions F with tail behaviour $\overline{F}(x) = (1 + o(1))\lambda x^{-\gamma}, \lambda, \gamma \in (0, \infty)$ as $x \to \infty$.
(d) When $\mathbf{X}$ is a W_p scale mixture random vector, then $\mathbf{X}/c, c \in (0, \infty)$ is also a W_p scale mixture random vector. Further if $R \sim F$ with F in the max-domain of attraction of Ψ_γ, then also R/c has distribution function in the max-domain of attraction of Ψ_γ. Hence the extension of our asymptotic results in (4.3) and (4.4) for F with upper endpoint $x_F \in (0, \infty) \setminus \{1\}$ follows easily.

5 Asymptotics of Concomitants of Order Statistics

We present next an application of the Kotz approximation concerning the asymptotic distribution of concomitants of order statistics from a bivariate sample with underlying W_p scale mixture distribution. Let therefore $(X_j, X_j), j = 1, \ldots, n$ be independent bivariate random vectors with stochastic representation (2.2). The ith concomitant of order statistics $Y_{[i:n]}$ is defined as follows: if we order the pairs based on the order statistics $X_{1:n} \leq \cdots \leq X_{n:n}$, then $Y_{[i:n]}$ is the second component of the pair with first component the ith order statistics $X_{i:n}$.

The main applications of concomitants are in selection procedures, ranked-set sampling, prediction analysis, and inference problems, see e.g., David and Nagaraja (2003), or Wang (2008) for details.

In an asymptotic context, Nagaraja and David (1994) derive interesting results for the maximum of the concomitants of order statistics $Y_{n,k} := \max_{1 \leq i \leq k} Y_{[n-i+1:n]}, n > 1$, with k a fix integer. Related asymptotic results can be found in Eddy and Gales (1981), Galambos (1987), Joshi and Nagaraja (1995), and Ledford and Tawn (1998).

In fact, by Result 1 of Nagaraja and David (1994) and the Kotz approximation the asymptotic distribution of $Y_{n,k}, n \to \infty$ follows easily. Below we shall investigate the asymptotic behaviour of $(Y_{[n:n]},, \ldots, Y_{[n-k+1:n]}), 1 \leq k < n$. Since the joint distribution function of $(Y_{[n:n]}, \ldots, Y_{[n-k+1:n]}), 1 \leq k < n$ conditioning on the order statistics $(X_{n:n} = x_1, \ldots, X_{n-k+1:n} = x_k)$ is

$$\prod_{i=1}^{k} \boldsymbol{P}\{Y_1 \leq y_i | X_1 = x_i\}, \quad x_i, y_i \in I\!R \tag{5.1}$$

we can apply the Kotz approximation developed in Section 3. More specifically, we shall show that $(Y_{[n:n]},, \ldots, Y_{[n-k+1:n]})$ are asymptotically independent, provided that the W_p scale mixture random vector (X_1, Y_1) satisfies the assumptions of Theorem 3.1. It is interesting that in this model the limit distribution of the concomitants of order statistics depends only on α and p.

Theorem 5.1. *Let $(X_j, Y_j), j = 1, \ldots, n$ be a random sample of W_p scale mixture random vectors as defined in (2.2). Assume that $p \in (1, \infty), R \sim F$, with F in the Gumbel max-domain of attraction with some scaling function w, and $W_p \asymp g$. Define A_n, B_n by*

$$A_n := (1 - \rho^p)^{1/p} \frac{b_n}{(b_n w(b_n))^{1/p}}, \quad B_n := \rho b_n, \quad b_n := H^{-1}(1 - 1/n), \quad n > 1,$$

with H^{-1} the inverse of the distribution function of X_1. If (3.3) holds with $\alpha \in (0, \infty)$, and the positive density g satisfies (3.4), then for any $k \geq 1$ we have

$$\left(\frac{Y_{[n:n]} - B_n}{A_n}, \ldots, \frac{Y_{[n-k+1:n]} - B_n}{A_n} \right) \quad \overset{d}{\to} \quad (\eta_1, \ldots, \eta_k), \quad n \to \infty, \tag{5.2}$$

where $\eta_1, \ldots, \eta_k$ are independent random variables being symmetric about 0 satisfying $|\eta_i|^p \sim \Gamma(\alpha, 1/p), i \leq n$.

In Theorem 5.1 we discuss only the case $p \in (1, \infty)$. When $p \in (0, 1]$ and $\rho = 0$ the same asymptotic result holds.

If (X_1, Y_1) is a standard bivariate Gaussian random vector, then (5.2) implies

$$\left(\frac{Y_{[n:n]} - \rho\sqrt{2 \ln n}}{\sqrt{1 - \rho^2}}, \ldots, \frac{Y_{[n-k+1:n]} - \rho\sqrt{2 \ln n}}{\sqrt{1 - \rho^2}} \right) \quad \overset{d}{\to} \quad (\eta_1, \ldots, \eta_k), \quad n \to \infty, \tag{5.3}$$

with $\eta_i, i \leq n$ independent standard Gaussian random variables. Hence for any $k \geq 1$ we obtain (see Nagaraja and David (1994), Joshi and Nagaraja (1995))

$$\frac{Y_{n,k} - \rho\sqrt{2 \ln n}}{\sqrt{1 - \rho^2}} \quad \overset{d}{\to} \quad \max_{1 \leq i \leq k} \eta_i, \quad n \to \infty. \tag{5.4}$$

Further, remark that under the assumptions of Theorem 5.1

$$\left(\frac{Y_{[n-i+1:n]} - B_n}{A_n}, \frac{Y_{n-i+1:n} - b_n}{a_n}\right) \xrightarrow{d} (\eta_i, \xi_i), \quad n \to \infty, \tag{5.5}$$

where η_i is independent of $\xi_i, i \geq 1$ and $\xi_i \sim \Lambda'(-\ln \Lambda)^i/i!$ with Λ' the density of Λ.

6   Proofs

We present next a lemma and then proceed with the proofs.

Lemma 6.1. *Let $g : [0,1] \to (0, \infty)$ be a positive measurable function, and let F be a distribution function on $[0, \infty)$ with upper endpoint $x_F \in (0, \infty]$.*

a) Let $y \in (0, \infty), z \in [y, \infty)$ be given constants. If F is in the max-domain of attraction of $\Phi_\gamma, \gamma \in (0, \infty)$ and $g(y/r) \leq cr^\delta, \forall r \geq z$, with $\delta < \gamma + 1$ and $c \in (0, \infty)$, then we have

$$\int_{uz}^\infty g(uy/r)\frac{1}{r}\, dF(r) \quad = \quad (1 + o(1))\overline{F}(u)\gamma \int_z^\infty g(y/r)r^{-\gamma-2}\, dr, \quad u \to \infty. \tag{6.1}$$

In the special case g is a density function of a positive random variable Z, then (6.1) holds, provided that $\mathbf{E}\{Z^\tau\} \in (0, \infty)$ for some $\tau \in (\gamma, \infty)$.

b) Suppose that F satisfies (4.1). If (3.3) holds with $\alpha \in (0, \infty)$, then for any $\beta \in \mathbb{R}$ and $\infty > z > y \geq 0$ we have

$$\int_{1-u(z-y)}^1 g((1-uz)/x)x^\beta\, dF(x) \quad = \quad (1+o(1))\gamma\overline{F}(1-u)g(1-u)\int_0^{z-y}(z-t)^{\alpha-1}t^{\gamma-1}\, dt, \quad u \downarrow 0. \tag{6.2}$$

c) If $F \in MDA(\Lambda, w)$, then for any given constant $\beta \in \mathbb{R}$

$$\lim_{u \to \infty} \frac{(uw(u))^\beta \overline{F}(\mu u)}{\overline{F}(u)} \quad = \quad 0, \quad \forall \mu \in (1, \infty) \tag{6.3}$$

and moreover, if (3.3) holds with $\alpha \in (0, \infty)$ and (3.4) is satisfied, then for any $z \in [0, \infty)$ we have

$$\frac{1}{\overline{F}(u)}\int_{u+z/w(u)}^{x_F} g(u/x)x^\beta\, dF(x) \quad = \quad (1+o(1))g(1-\frac{1}{uw(u)})u^\beta \int_z^\infty s^{\alpha-1}\exp(-s)\, ds, \quad u \uparrow x_F. \tag{6.4}$$

Proof. a) Assume for simplicity that F possesses a density function f which is also regularly varying. The general case follows applying Lemma 2 in Kaj et al. (2007). By the assumption on g the integral $I_{y,z} := \int_z^\infty g(y/r)\frac{1}{r}r^{-\gamma-1}\, dr$ is finite. Further, the regular variation of f implies

$$\lim_{u \to \infty} \frac{f(u)}{u\overline{F}(u)} = \gamma,$$

hence applying Karamata's Theorem (see e.g., Resnick (2008)) we may write

$$\int_{uz}^\infty g(uy/r)\frac{1}{r}\, dF(r) \quad = \quad \frac{f(u)}{u}\int_z^\infty g(y/r)\frac{1}{r}\frac{f(ur)}{f(u)}\, dr$$
$$= \quad (1+o(1))\gamma I_{y,z}\overline{F}(u), \quad u \to \infty.$$

If g is a density function of some positive random variable $Z \in (0,1)$, then $I_{y,z}$ is finite if $\mathbf{E}\{Z^\tau\} < \infty, \tau \in (\gamma, \infty)$, and the statement can be established with the same arguments as above.

b) Transforming the variables for any $u \in (0,1)$ we have

$$\frac{1}{\overline{F}(1-u)}\int_{1+uy-uz}^1 g((1-uz)/x)x^\beta\, dF(x) \quad = \quad \int_0^{y-z} g((1-uz)/(1-us))[1-us]^\beta\, dF(1-us)/\overline{F}(1-u).$$

Consequently, by the max-domain of attraction assumption on F and the regular variation of g at 1

$$\frac{1}{\overline{F}(1-u)} \int_0^{y-z} g((1-uz)/x)x^\beta \, dF(x) = (1+o(1))g(1-u)\gamma \int_0^{y-z} (z-s)^{\alpha-1} s^{\gamma-1} \, ds, \quad u \downarrow 0,$$

hence the claim follows.

c) Set $F_u(x) := F(u+x/w(u))/\overline{F}(u), v(u) := uw(u), u > 0, x \in \mathbb{R}$. The Gumbel max-domain of attraction assumption on F implies (see e.g. Resnick (2008))

$$\lim_{u \uparrow x_F} v(u) = \infty, \quad \lim_{u \uparrow x_F} w(u)(x_F - u) = \infty \text{ if } x_F < \infty \tag{6.5}$$

and

$$\lim_{u \uparrow x_F} F_u(s) - F_u(t) = \exp(-t) - \exp(-s), \quad s, t \in \mathbb{R}.$$

Transforming the variables for u large

$$\frac{1}{\overline{F}(u)} \int_{u+z/w(u)}^{x_F} g(u/x)x^\beta \, dF(x) = u^\beta \int_z^{w(u)(x_F-u)} g(1/[1+x/v(u)])[1+x/v(u)]^\beta \, dF_u(x)$$

$$= (1+o(1))u^\beta g(1-1/v(u)) \int_z^\infty (1+o(1))x^{\alpha-1} dF_u(x).$$

We consider only the case $x_F = \infty$ and omit the proof when $x_F \in (0, \infty)$ since it can be established with the same arguments. For any $\varepsilon > 0$ we may write

$$\int_{u+z/w(u)}^\infty g(u/x)x^\beta \, dF(x) = \int_{u+z/w(u)}^{(1+\varepsilon)u} g(u/x)x^\beta \, dF(x) + \int_{(1+\varepsilon)u}^\infty g(u/x)x^\beta \, dF(x) =: I_\varepsilon(u) + J_\varepsilon(u).$$

As in the proof of Lemma 3.5 in Hashorva (2006) utilising further Potter's upper bound (see De Haan and Ferreira (2006) or Resnick (2008)) for the regularly varying function g, for any $\varepsilon > 0$ sufficiently small we obtain (recall $\lim_{u \to \infty} v(u) = \infty$)

$$\lim_{u \to \infty} \frac{I_\varepsilon(u)}{u^\beta g(1-1/v(u))} = \lim_{u \to \infty} \int_z^{(1+\varepsilon)v(u)} (1+o(1))x^{\alpha-1} dF_u(x) = \int_z^\infty x^{\alpha-1} \exp(-x) \, dx.$$

By Lemma 4.5 in the aforementioned paper

$$I_{\alpha,u} := \int_u^\infty (x^2 - u^2)^\alpha \, dF(x) = (1+o(1))\Gamma(\alpha+1)\left(\frac{2}{v(u)}\right)^\alpha u^{2\alpha}\overline{F}(u), \quad u \to \infty.$$

Since for any $\xi > 1$ we have $\lim_{u \to \infty} \overline{F}(\xi u)/F(u) = 0$, and furthermore

$$I_{\alpha,u} \geq \int_{\xi u}^{2\xi u} (x^2 - u^2)^\alpha \, dF(x) \geq (1+o(1))\xi^2 u^{2\alpha}\overline{F}(\xi u), \quad u \to \infty$$

(6.3) follows easily. Next, by (3.4) and (6.3)

$$J_\varepsilon(u) \leq c_\varepsilon u^{\gamma_\varepsilon} \int_{(1+\varepsilon)u}^\infty x^{\beta-\gamma_\varepsilon} \, dF(x) = o(I_\varepsilon(u)), \quad u \to \infty,$$

thus the result follows. $\qquad \square$

PROOF OF THEOREM 3.1 a) Let $R_n, n \geq 1$ be random variable with survivor function

$$\boldsymbol{P}\{R_n > s\} := \frac{\int_{(\tau_n^p+s^p)^{1/p}}^{x_F} g(\tau_n/x)\frac{1}{x} \, dF(x)}{\int_{\tau_n}^{x_F} g(\tau_n/x)\frac{1}{x} \, dF(x)}, \quad \forall s \in (0, (x_F{}^p - \tau_n^p)^{1/p}), \tag{6.6}$$

where $\tau_n := \|\boldsymbol{u}_{n,J}\|_A, n \geq 1$. By (6.5) and Lemma 6.1 for any $z > 0$ we have

$$\lim_{n \to \infty} \boldsymbol{P}\{h_n R_n > z\} = \lim_{n \to \infty} \frac{\int_{\tau_n+(1+o(1))z^p/(pw(\tau_n))}^{x_F} g(\tau_n/x)\frac{1}{x} \, dF(x)}{\int_{\tau_n}^{x_F} g(\tau_n/x)\frac{1}{x} \, dF(x)}$$

$$
\begin{aligned}
&= \frac{\int_{z^p/p}^{\infty} t^{\alpha-1}\exp(-t)\,dt}{\int_0^{\infty} t^{\alpha-1}\exp(-t)\,dt}\\
&= \boldsymbol{P}\{\mathcal{R}_\alpha > z\},
\end{aligned}
$$

with $\mathcal{R}_\alpha$ a positive random variable satisfying $\mathcal{R}_\alpha^p \sim \Gamma(\alpha, 1/p)$. Since $\mathcal{R}_\alpha$ is independent of $\boldsymbol{U}_I$ the first claim follows using further (2.4).

b) As in Hashorva (2007c) it follows that X_d has distribution function H in the Gumbel max-domain of attraction with the scaling function w. Our proof below is quite similar to the proof of Theorem 3.1 in Hashorva (2008a), therefore we omit some details. Next, define $h_n := (u_n w(u_n))^{(1-p)/p} w(u_n), n \geq 1$, and let $\boldsymbol{x} \in \mathbb{R}^d, t \in \mathbb{R}$ be given. By (6.5) $\lim_{n\to\infty} h_n/w(u_n) = 0$ holds if $p > 1$, and $\lim_{n\to\infty} h_n/w(u_n) = 1$ when $p = 1$. Since further (see e.g., Resnick (2008))

$$
\lim_{u\uparrow x_F} \frac{w(u + z/w(u))}{w(u)} = 1 \tag{6.7}
$$

uniformly for z in compact sets of $\mathbb{R}$ for any $p \in (0,\infty)$ and A_{IJ} with elements equal 0 when $p \in (0,1)$ we have

$$
\lim_{n\to\infty} \boldsymbol{P}\Big\{h_n(\boldsymbol{X}_I - u_n A_{IJ}) \leq \boldsymbol{x}_I \big| X_d = u_n + t/w(u_n)\Big\} = \boldsymbol{P}\{A_{II}\mathcal{R}_\alpha \boldsymbol{U}_I + 1_p t A_{IJ} \leq \boldsymbol{x}_I\}
$$

locally uniformly for $t \in \mathbb{R}$ with $1_p := 1$ if $p = 1$ and $1_p := 0$ otherwise. Hence along the lines of the proof of Theorem 3.3 in Hashorva (2006) for any $x_d > 0$ we obtain

$$
\begin{aligned}
&\boldsymbol{P}\Big\{h_n(\boldsymbol{X}_I - u_n A_{IJ}) \leq \boldsymbol{x}_I, w(u_n)(X_d - u_n) \leq x_d \big| X_d > u_n\Big\}\\
&= \int_0^{x_d} \boldsymbol{P}\Big\{h_n(\boldsymbol{X}_I - u_n A_{IJ}) \leq \boldsymbol{x}_I \big| X_d = u_n + t/w(u_n)\Big\}\, dH(u_n + t/w(u_n))/\overline{H}(u_n)\\
&\to \int_0^{x_d} \boldsymbol{P}\{A_{II}\mathcal{R}_\alpha \boldsymbol{U}_I + 1_p t A_{IJ} \leq \boldsymbol{x}_I\}\exp(-t)\, dt, \quad n \to \infty,
\end{aligned}
$$

thus the proof is complete. $\qquad\square$

PROOF OF THEOREM 4.1 a) Let $R_n, n \geq 1$ be positive random variables as in (6.6) corresponding to $\boldsymbol{u}_{n,J}$. Since $\lim_{n\to\infty} a_n = 0$, by the assumptions on F and g for any $t \in (0,1)$ we obtain (set $\overline{a}_n := 1 - a_n, h_n := (pa_n)^{-1/p}, n \geq 1$)

$$
\begin{aligned}
\lim_{n\to\infty} \boldsymbol{P}\{h_n R_n > t\} &= \lim_{n\to\infty} \frac{\int_{[\overline{a}_n^p + pa_n t^p]^{1/p}}^{1} g(\overline{a}_n/r)\frac{1}{r}\, dF(r)}{\int_{\overline{a}_n}^{1} g(\overline{a}_n/r)\frac{1}{r}\, dF(r)}\\
&= \lim_{n\to\infty} \frac{\int_{1-a_n(1-t^p)(1+o(1))}^{1} g(\overline{a}_n/r)\frac{1}{r}\, dF(r)}{\int_{\overline{a}_n}^{1} g(\overline{a}_n/r)\frac{1}{r}\, dF(r)}\\
&= \frac{\int_0^{1-t^p}(1-x)^{\alpha-1}x^{\gamma-1}\, dx}{\int_0^1 (1-x)^{\alpha-1}x^{\gamma-1}\, dx} = 1 - B(t^p, \alpha, \gamma),
\end{aligned}
$$

where $B(x, \alpha, \gamma), x \in (0,1)$ is the Beta distribution function with positive parameters α, γ implying thus

$$
h_n R_n \xrightarrow{d} \mathcal{R}_{\alpha,\gamma}, \quad n \to \infty,
$$

with $\mathcal{R}_{\alpha,\gamma} > 0$ such that $\mathcal{R}_{\alpha,\gamma}^p \sim Beta(\alpha, \gamma)$ being independent of $\boldsymbol{U}_I$. Consequently,

$$
h_n A_{II} R_n \boldsymbol{U}_I \xrightarrow{d} A_{II}\mathcal{R}_{\alpha,\gamma}\boldsymbol{U}_I, \quad n \to \infty,
$$

hence the first claim follows.

b) In view of Theorem 3.1 in Hashorva (2008c) the random variable X_d has distribution function H in the max-domain of attraction of $\Psi_{\alpha+\gamma}$. If $\boldsymbol{u}_{n,J} = 1 - a_n t, n \geq 1, t > 0$, then for any $\boldsymbol{x} \in \mathbb{R}^d$ and $p \in (0,\infty)$ locally uniformly for $t > 0$

$$
\boldsymbol{P}\Big\{\boldsymbol{X}_I \leq A_{IJ} + (pa_n)^{1/p}\boldsymbol{x}_I \big| X_d = 1 - a_n t\Big\} \to \boldsymbol{P}\Big\{A_{II}\mathcal{R}_{\alpha,\gamma}\boldsymbol{U}_I - 1_p A_{IJ} \leq t^{-1/p}\boldsymbol{x}_I\Big\}, \quad n \to \infty
$$

holds. Hence for any $x_d \in (-\infty, 0)$

$$P\Big\{ \boldsymbol{X}_I \le A_{IJ} + (pa_n)^{1/p}\boldsymbol{x}_I, (X_d - 1)/a_n \le x_d \big| X_d > 1 - a_n \Big\}$$

$$= \int_{-1}^{x_d} P\Big\{ \boldsymbol{X}_I \le A_{IJ} + (pa_n)^{1/p}\boldsymbol{x}_I \big| X_d = 1 + a_n t \Big\} \, dH(1 + a_n t)/\overline{H}(1 - a_n)$$

$$\to \int_{-1}^{x_d} P\{A_{II}\mathcal{R}_{\alpha,\gamma}\boldsymbol{U}_I - 1_p A_{IJ} \le |t|^{-1/p}\boldsymbol{x}_I\} \, d(1 - |t|^{\alpha+\gamma}), \quad n \to \infty$$

$$= P\{|\mathcal{E}|^{1/p}(A_{II}\mathcal{R}_{\alpha,\gamma}\boldsymbol{U}_I - 1_p A_{IJ}) \le \boldsymbol{x}_I, \mathcal{E} \le x_d\},$$

with $\mathcal{E}$ a negative random variable with distribution function $M(s) := 1 - |s|^{\alpha+\gamma}, s \in [-1, 0]$. Thus the result follows.

c) Set $\boldsymbol{u}_n := \boldsymbol{a}/a_n, n \ge 1, c := \|\boldsymbol{a}_J\|_A$ and define a positive random variable $R_n, n \ge 1$ as in (6.6). Since W_p is bounded for any $\lambda \in (0, \infty)$ we have $\boldsymbol{E}\{W_p^\lambda\} \in (0, \infty)$. Hence for any $z > 0$ Lemma 6.1 implies

$$
\begin{aligned}
\lim_{n\to\infty} \boldsymbol{P}\{R_n > a_n z\} &= \lim_{n\to\infty} \frac{\int_{a_n(c^p + z^p)^{1/p}}^\infty g(a_n c/r)\frac{1}{r}\, dF(r)}{\int_{a_n c}^\infty g(a_n c/r)\frac{1}{r}\, dF(r)} \\
&= \frac{\int_{(c^p + z^p)^{1/p}}^\infty g(c/r)r^{-\gamma-2}\, dr}{\int_c^\infty g(c/r)r^{-\gamma-2}\, dr} \\
&= \boldsymbol{P}\{\mathcal{R} > z\},
\end{aligned}
$$

where $\mathcal{R} \sim Q_{M,g,c}$ being independent of $\boldsymbol{U}_I$ and $M(x) := 1 - c^\gamma x^{-\gamma}, \forall x \ge c > 0$. Thus the result follows easily. □

PROOF OF THEOREM 5.1 The proof can be established by extending Result 1 of Nagaraja and David (1994) to the higher dimensional setup and utilising further the Kotz approximation.

We give next the sketch of another proof. For notational simplicity assume that $k = 2$. Define next $b_n := H^{-1}(1 - 1/n), a_n := 1/w(b_n), n \ge 1$ and let A_n, B_n be as in (5.2). In view of (5.1) for $n > 1$ we have

$$P\Big\{ \frac{Y_{[n:n]} - B_n}{A_n} \le y_1, \frac{Y_{[n-1:n]} - B_n}{A_n} \le y_2 \Big\} = \int_{x_1 > x_2} \prod_{i=1}^2 P\{Y_1 \le A_n y_i + B_n | X_1 = a_n x_i + b_n\} \, dD_n^*(x_1, x_2),$$

with $D_n^*(x_1, x_2) = \boldsymbol{P}\{X_{n:n} \le a_n x_1 + b_n, X_{n-1:n} \le a_n x_2 + b_n\}$. As in the proof of Theorem 3.1 we have that the distribution function H of X_1 is in the Gumbel max-domain of attraction with scaling function w, implying thus the joint convergence of upper order statistics (see e.g., Falk et al. (2004)), i.e.,

$$\lim_{n\to\infty} D_n^*(x_1, x_2) = \mathcal{D}(x_1, x_2), \quad \forall x_1, x_2, \quad x_1 > x_2,$$

with $\mathcal{D}$ a bivariate distribution function. By the assumptions and the properties of the scaling function w

$$\prod_{i=1}^2 P\{Y_1 \le A_n y_i + B_n | X_1 = a_n x_i + b_n\} \to \prod_{i=1}^2 P\{I_2\mathcal{R}_\alpha \le y_i\}, \quad n \to \infty$$

holds locally uniformly for $x_1, x_2 \in \mathbb{R}$. Hence with similar arguments as in the proof of Theorem 4.1 in Hashorva (2008b) we obtain

$$\lim_{n\to\infty} \boldsymbol{P}\Big\{ \frac{Y_{[n:n]} - B_n}{A_n} \le y_1, \frac{Y_{[n-1:n]} - B_n}{A_n} \le y_2 \Big\} = \prod_{i=1}^2 P\{I_2\mathcal{R}_\alpha \le y_i\} \int_{x_1 > x_2} d\mathcal{D}(x_1, x_2) = \prod_{i=1}^2 P\{I_2\mathcal{R}_\alpha \le y_i\},$$

thus the result follows. □

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